A PROOF OF THE DVORETZKY-ROGERS THEOREM

BY

IVAN SINGER

ABSTRACT

We give a new proof of the famous Dvoretzky-Rogers theorem ([2], Theorem 1), according to which a Banach space E is finite-dimensional if every unconditionally convergent series in E is absolutely convergent.

It is clearly sufficient to prove the theorem for a separable Banach space E. Now, if every unconditionally convergent series in E is absolutely convergent, then every continuous linear mapping u of E into $(l^1)^* = m$ is integral(*). Since E is separable, let u be an isometrical embedding ([1], Theorem a)) of E into m. Then the "astriction" $v = u: E \to u(E)$ of u is also integral, whence there exist(**) a Hilbert space H and continuous linear mappings $w_1: E \to H$, $w_2: H \to u(E)$ such that $v = w_2 w_1$. Since v is onto, w_2 must be onto, whence the isomorphisms (linear homeomorphisms)

$$E \sim u(E) \sim H/Ker\, w_2 \sim H_1 \ ,$$

where H_1 is a Hilbert space. Thus, by our hypothesis, every unconditionally convergent series in H_1 is absolutely convergent. It is trivial that in this case we have $\dim H_1 < \infty$ (since e.g. the series $\sum_{n=1}^{\infty} x_n$ in l^2, where

$$x_n = \Big\{\underbrace{0, \cdots, 0}_{n-1}, \frac{1}{n}, 0, \cdots\Big\}, \qquad n = 1, 2, \cdots$$

is unconditionally convergent, but not absolutely convergent), whence also $\dim E < \infty$. This completes the proof.

REMARK. Our proof is, in a certain sense, dual to that of Grothendieck ([3], pp. 149–150). In fact, he considers, *for every sequence* $\{f_n\} \subset E^*$ *with* $\|f_n\| \leqq 1$, the mapping $w: l_1 \to E^*$ defined by

$$w(\{\lambda_n\}) = \sum_{i=1}^{\infty} \lambda_i f_i \qquad (\{\lambda_n\} \in l^1),$$

while in the above proof we consider the isometry $u: E \to m$ defined by

$$u(x) = \{f_n(x)\} \qquad (x \in E),$$

Received February 7, 1965.

* See e.g. [3], p. 148, Lemma 15; for an elementary proof, see [4], p. 162.

** See e.g. [3], p. 163; for an elementary proof, see [4], p. 160, proposition 2 (this latter proof is valid only for real spaces and mappings u with values in conjugate spaces, but it is easy to adapt it to the general case).

where $\{f_n\} \subset E^*$ *is a fixed* $\sigma(E^*, E)$*-dense sequence in* $\{f \in E^* \mid \|f\| \leqq 1\}$ (see [1], the proof of theorem a)), and for such $\{f_n\}$ it is clear that $w^*|_E = u$. However, the proof of Grothendieck makes use of two results on integral operators(*) and of the Eberlein theorem on reflexivity, while the above proof is perhaps slightly more simple.

References

1. S. Banach and S. Mazur, *Zur Theorie der linearen Dimension*, Studia Math., **4** (1933), 100–112.
2. A. Dvoretzky and C. A. Rogers, *Absolute and unconditional convergence in normed linear spaces*, Proc. Nat. Acad. Sci., **36** (1950), 192–197.
3. A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc., no. 16., (1955).
4. A. Pełczyński, *A proof of the theorem of Grothendieck on the characterization of nuclear spaces* (Russian), Prace Mat., 7 (1962), 155–167.

Institut de Mathematique,
Academie de la R. P. Roumaine

* Namely, of: (1) every integral operator is weakly compact ([3], p. 131, theorem 9, 10 and (2) every integral mapping into a reflexive space is compact ([3], p. 134, Corollary 2).